

## ON ADMISSIBLE STRATEGIES IN ROBUST UTILITY MAXIMIZATION

KEITA OWARI

*Graduate School of Economics, The University of Tokyo*  
 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

The existence of optimal strategy in robust utility maximization is addressed when the utility function is finite on the entire real line. A delicate problem in this case is to find a “good definition” of admissible strategies to admit an optimizer. Under certain assumptions, especially a kind of time-consistency property of the set  $\mathcal{P}$  of probabilities which describes the model uncertainty, we show that an optimal strategy is obtained in the class of those whose wealths are supermartingales under all local martingale measures having a finite generalized entropy with one of  $P \in \mathcal{P}$ .

## 1. INTRODUCTION

This paper analyzes a *qualitative aspect* of the problem of robust utility maximization. Given a utility function  $U$  and a set  $\mathcal{P}$  of probabilities which describes the model uncertainty, the basic problem of this paper is to maximize the *robust utility functional*

$$X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)]$$

over all terminal wealths  $x + \theta \cdot S_T = x + \int_0^T \theta dS$  of admissible strategies  $\theta$ , where  $S$  is an underlying semimartingale. When  $U$  is finite only on the positive half-line, the duality theory for this problem in the spirit of [15, 16] has been studied in both *quantitative* and *qualitative* aspects (e.g. [23], [22], [8]). In the case of utility taking finite values for all  $x \in \mathbb{R}$ , [18] shows the key duality, while [8] and [17] give a *partial result* on the existence of optimal strategy which we shall complete in this paper. See also [9] for more comprehensive reference and the background of the robust utility maximization problem.

A key subtlety intrinsic to the case of utility on  $\mathbb{R}$  is the “good definition” of *admissible strategies*  $\theta$ , which will constitute the central theme of this paper. In this case, a universal and conceptually natural definition of admissibility is that  $\theta \cdot S$  is uniformly bounded from below by some constant, which completely determines the quantitative nature of the problem. This class, however, typically fails to admit an optimizer. On the other hand, if  $U$  is  $-\infty$  on  $\mathbb{R}_-$ , the only natural (non-redundant) definition of admissibility is that the stochastic integral  $\theta \cdot S$  is bounded from below by  $-x$ , and an optimal strategy is indeed obtained in this class under certain mild assumptions (see [23, 22]).

In the classical case (i.e.,  $\mathcal{P} = \{P\}$ , say), the question of the good definition of admissibility is closely analyzed by [21] following the observation by [7] and [14] in the case of

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*E-mail address:* [keita.owari@gmail.com](mailto:keita.owari@gmail.com).

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exponential utility. [21] shows that a “good definition” which yields us an optimal strategy is that  $\theta \cdot S$  is a supermartingale under all local martingale measures  $Q$  which has a “finite entropy” with the physical probability  $P$ . We denote the class of such  $\theta$  by  $\Theta_V(P)$  (see Section 2 for precise definitions including the meaning of “finite entropy”). Note that this class contains the usual admissible class, and the supermartingale property is consistent to the “No-Arbitrage philosophy”. Thus  $\Theta_V(P)$  is acceptably natural choice *when a single physical probability is specified*.

In the general robust case with  $\mathcal{P}$  containing (infinitely) many elements, [8] (see also [17] for a slight generalization) provides a partial analogue of the above result which states that, under certain stronger assumptions, an optimal strategy is obtained in the class of  $\theta$  with  $\theta \cdot S$  being a supermartingale under all local martingale measures  $Q$  having a finite entropy w.r.t. a certain element  $\hat{P} \in \mathcal{P}$  called a *least favorable measure*, i.e., in the class  $\Theta_V(\hat{P})$ . Here a dissatisfaction comes of course from the dependence of admissibility on  $\hat{P}$ . In *philosophy*,  $\mathcal{P}$  is the set of *candidates* of real world models, and we do not know which one is true. Thus an “admissible strategy” should be *universally* admissible for all candidates  $P \in \mathcal{P}$ . Also, the least favorable probability  $\hat{P}$  is a part of solution to the dual problem of robust utility maximization, hence the class  $\Theta_V(\hat{P})$  is not *a priori* available.

In this view, a seemingly natural admissible class is  $\bigcap_{P \in \mathcal{P}} \Theta_V(P)$  which is universal and contains all  $\theta$  whose stochastic integrals are bounded below. Thus our central question in this paper is:

**Question 1.** Does the class  $\bigcap_{P \in \mathcal{P}} \Theta_V(P)$  admit an optimal strategy?

The main result (Theorem 3.2) states that this is indeed the case if (in addition to standard assumptions) the set  $\mathcal{P}$  of candidate models has a *time-consistency* property. We proceed as follows. The first step is to construct a so-called “optimal claim” for the abstract version of robust utility maximization, from which a candidate of optimal strategy  $\hat{\theta}$  is derived through a predictable representation argument. This part is mostly standard excepting some technicality, but we give a slightly better description of optimal claim. Note that the additional time-consistency assumption is not required at this stage. The crucial step is to verify the supermartingale property of  $\hat{\theta} \cdot S$  under all local martingale measures  $Q$  which has a finite entropy with *some*  $P \in \mathcal{P}$  but *its entropy with  $\hat{P}$  is infinite*. We shall do this by a (slight surprisingly) simple trick.

## 2. FORMULATION

We fix a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  as well as a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$  satisfying *the usual conditions*, where  $T \in (0, \infty)$  is a fixed time horizon. Though many probabilities on  $(\Omega, \mathcal{F})$  will appear in the sequel, the probability  $\mathbb{P}$  plays the role of reference probability, i.e., every probabilistic notion is defined under  $\mathbb{P}$  unless other probability is explicitly specified as  $E_P[\cdot]$ ,  $L^1(P)$  etc. In particular, the underlying asset prices  $S$  is a  $d$ -dimensional  $\mathbb{P}$ -càdlàg semimartingale, and we assume:

(A1)  $S$  is  $\mathbb{P}$ -locally bounded.

Let  $\mathcal{P}$  be a set of probabilities  $P \ll \mathbb{P}$ , which we can (and do) embed into  $L^1$  via the mapping  $P \mapsto dP/d\mathbb{P}$ . In this sense, we assume:

(A2)  $\mathcal{P}$  is convex and  $\sigma(L^1, L^\infty)$ -compact.

We work with a utility function  $U : \mathbb{R} \rightarrow \mathbb{R}$  which we assume

(A3)  $U$  is differentiable, strictly concave on  $\mathbb{R}$ , and  $U'(-\infty) = \infty$ ,  $U'(\infty) = 0$ ,

and satisfies the condition of *reasonable asymptotic elasticity*:

$$(A4) \quad \liminf_{x \rightarrow -\infty} \frac{xU'(x)}{U(x)} > 1 \text{ and } \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

The conjugate of utility function  $U$  is denoted by  $V$ , i.e.,

$$(2.1) \quad V(y) := \sup_{x \in \mathbb{R}} (U(x) - xy), \quad y \in \mathbb{R}.$$

The assumptions (A3) and (A4) guarantee that  $V$  is a “nice” convex function (see [10], [19] for details). Using this function, we introduce a generalized entropy:

$$(2.2) \quad V(\nu|P) := \begin{cases} E_P[V(d\nu/dP)] & \text{if } \nu \ll P, \\ +\infty & \text{otherwise} \end{cases}$$

for any positive finite measure  $\nu \ll \mathbb{P}$  and  $P \in \mathcal{P}$ . When  $U(x) = 1 - e^{-x}$  (exponential utility) and  $Q$  is a probability with  $Q \ll P$ , we have  $V(Q|P) = E_Q[\log(dQ/dP)]$ , i.e., the relative entropy. Abusing the terminology, we still call the map  $V(\cdot|\cdot)$  the *generalized entropy* associated to  $V$ . We define also the *robust generalized entropy* by

$$(2.3) \quad V(Q|\mathcal{P}) := \inf_{P \in \mathcal{P}} V(Q|P) < \infty.$$

Let  $\mathcal{M}_{loc}$  be the set of all local martingale measures for  $S$ , i.e., probabilities  $Q \ll \mathbb{P}$  under which  $S$  is a local martingale. We then set

$$(2.4) \quad \mathcal{M}_V := \{Q \in \mathcal{M}_{loc} : V(Q|\mathcal{P}) < \infty\}.$$

Generically, for any set  $\mathcal{Q}$  of probabilities  $Q \ll \mathbb{P}$ , we denote by  $\mathcal{Q}^e$  the set of  $Q \in \mathcal{Q}$  with  $Q \sim \mathbb{P}$ . We assume the existence of *equivalent* local martingale measure with finite entropy in the following sense:

$$(A5) \quad \mathcal{M}_V^e := \{Q \in \mathcal{M}_V : Q \sim \mathbb{P}\} \neq \emptyset.$$

In particular, this implies the existence of  $(Q, P) \in \mathcal{M}_V^e \times \mathcal{P}$  such that  $Q \sim P \sim \mathbb{P}$  and  $V(Q|P) < \infty$ . See [17] for detail and other consequences of these assumptions.

Let  $L(S)$  be the totality of all  $(S, \mathbb{P})$ -integrable  $d$ -dimensional predictable processes,  $L_0(S) := \{\theta \in L(S) : \theta_0 = 0\}$ , and we denote by  $\theta \cdot S$  the stochastic integral of  $\theta \in L(S)$  w.r.t.  $S$ . See e.g., [12] or [13] for more information. When the utility function is finite on the entire real line, a conceptually natural choice of  $\Theta$  is

$$(2.5) \quad \Theta_{bb} := \{\theta \in L_0(S) : \theta \cdot S_0 = 0, \theta \cdot S \text{ is bounded from below}\}.$$

Then the value function of the robust utility maximization problem is given by

$$(2.6) \quad u(x) := \sup_{\theta \in \Theta_{bb}} \inf_{P \in \mathcal{P}} E_P[U(x + \theta \cdot S_T)], \quad x \in \mathbb{R}.$$

When we seek an optimal strategy, however, the class  $\Theta_{bb}$  is typically too small to admit an optimal strategy. We thus have to enlarge the admissible class. Our choice is the following.

$$(2.7) \quad \Theta_V := \{\theta \in L_0(S) : \theta \cdot S \text{ is a } Q\text{-supermartingale, } \forall Q \in \mathcal{M}_V\}.$$

**Remark 2.1** (Another equivalent formulation). We have defined the classes  $\mathcal{M}_V$  and  $\Theta_V$  through the *robust* generalized entropy  $Q \mapsto V(Q|\mathcal{P})$ . But the following equivalent formulation is sometimes useful for comparison. For each  $P \in \mathcal{P}$ , we set

$$(2.8) \quad \mathcal{M}_V(P) := \{Q \in \mathcal{M}_{loc} : V(Q|P) < \infty\},$$

$$(2.9) \quad \Theta_V(P) := \{\theta \in L_0(S) : \theta \cdot S \text{ is a } Q\text{-supermartingale } \forall Q \in \mathcal{M}_V(P)\}.$$

When a single  $P \in \mathcal{P}$  is fixed as the physical probability, the class  $\Theta_V(P)$  is shown to be an appropriate domain of utility maximization in [21]. Recalling (2.3), our choices  $\mathcal{M}_V$  and  $\Theta_V$  are rewritten respectively as

$$\mathcal{M}_V = \bigcup_{P \in \mathcal{P}} \mathcal{M}_V(P), \quad \Theta_V = \bigcap_{P \in \mathcal{P}} \Theta_V(P).$$

Thus our definition (2.7) is consistent to what we wrote in introduction.

Under the assumptions (A1) – (A5), a duality result (Theorem 2.3 of [18]) is applicable, which states in our case that for any  $\Theta$  with  $\Theta_{bb} \subset \Theta \subset \Theta_V$ , we have

$$(2.10) \quad u(x) = \sup_{\theta \in \Theta} \inf_{P \in \mathcal{P}} E_P[U(x + \theta \cdot S_T)] = \inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} (V(\lambda Q|\mathcal{P}) + \lambda x).$$

In particular, the value function is unchanged if we replace  $\Theta_{bb}$  by the larger class  $\Theta_V$ . Under the same assumptions, the right hand side, the *dual problem* of the (2.6), admits a solution  $(\hat{\lambda}, \hat{Q}) \in (0, \infty) \times \mathcal{M}_V$ , and the infimum  $V(\hat{\lambda} \hat{Q}|\mathcal{P}) = \inf_{P \in \mathcal{P}} V(\hat{\lambda} \hat{Q}|P)$  is attained by a  $\hat{P} \in \mathcal{P}$  since  $\mathcal{P}$  is weakly compact, and  $V(\cdot|\cdot)$  is lower semicontinuous. Thus the right hand side of (2.10) is also written as  $V(\hat{\lambda} \hat{Q}|\hat{P})$ , and we call the triplet  $(\hat{\lambda}, \hat{Q}, \hat{P})$  a dual optimizer.

A way of proving (2.10) and the existence of a solution  $(\hat{\lambda}, \hat{Q})$  is to closely analyze the robust utility functional  $X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)]$  on  $L^\infty$  characterizing  $V(\cdot|\mathcal{P})$  as its conjugate. Then the duality and the existence of  $(\hat{\lambda}, \hat{Q})$  follow *simultaneously* from Fenchel's duality theorem. See [18] for detail. Alternatively, one can separate the dual problem into the minimization of  $\lambda \mapsto \inf_{Q \in \mathcal{M}_V} V(\lambda Q|\mathcal{P}) + \lambda x$  and of  $Q \mapsto V(\lambda Q|\mathcal{P})$  for each  $\lambda$ . For the latter problem, called the *robust f-projection*, [8] proves the existence by establishing a uniform integrability criterion in terms of  $V(\cdot|\mathcal{P})$  in the spirit of the de la Vallée-Poussin theorem.

In contrast to the standard utility maximization, neither the uniqueness of  $(\hat{\lambda}, \hat{Q})$  (hence of the triplet  $(\hat{\lambda}, \hat{Q}, \hat{P})$ ) nor the equivalence  $\hat{Q} \sim \mathbb{P}$  hold in the robust case, as the following trivial example illustrates:

**Example 2.2.** Suppose  $\mathcal{M}_{loc}^e \neq \emptyset$ , and that  $\mathcal{M}_{loc}$  contains an element  $Q_0$  which is not equivalent to  $\mathbb{P}$ . Then we take  $\mathcal{P}$  so that  $Q_0 \in \mathcal{P} \subset \mathcal{M}_{loc}$ . In this case,  $\hat{\lambda}$  is uniquely determined as the minimizer of  $\lambda \mapsto V(\lambda) + \lambda x$ . Then a triplet  $(\hat{\lambda}, Q, P)$  is a dual optimizer if (and only if)  $P = Q \in \mathcal{P} \subset \mathcal{M}_{loc}$ . Indeed, by Jensen's inequality and the strict convexity of  $V$ ,  $V(\lambda Q|P) = E_P[V(\lambda dQ/dP)] \geq V(\lambda)$  whenever  $Q \ll P$ , and the “equality” holds if and only if  $Q = P$ . Hence  $(\hat{\lambda}, \hat{Q}, \hat{P})$  is not unique, and  $(\hat{\lambda}, Q_0, Q_0)$  is a solution with  $Q_0 \not\sim \mathbb{P}$ .

As for the equivalence, we still have  $\hat{Q} \sim \hat{P}$  whenever  $(\hat{\lambda}, \hat{Q}, \hat{P})$  is a dual optimizer (see [17], Theorem 2.7). Also, by an exhaustion argument, there exists a *maximal solution*  $(\hat{\lambda}, \hat{Q}, \hat{P})$  in the sense that if  $(\lambda, Q, P)$  is another dual optimizer, then  $P \ll \hat{P}$  (hence  $Q \ll \hat{Q}$ ) and  $\lambda dQ/dP = \hat{\lambda} d\hat{Q}/d\hat{P}$ ,  $P$ -a.s., where the density  $d\hat{Q}/d\hat{P}$  is defined  $\mathbb{P}$ -a.s. in the sense of *Lebesgue decomposition*. In particular, if  $(\hat{\lambda}, \hat{Q}, \hat{P})$  and  $(\tilde{\lambda}, \tilde{Q}, \tilde{P})$  are two maximal solution, then

$$(2.11) \quad \tilde{\lambda} d\tilde{Q}/d\tilde{P} = \hat{\lambda} d\hat{Q}/d\hat{P}, \quad \mathbb{P}\text{-a.s.},$$

See [17, Theorem 2.5 and Proposition 4.7]. This uniqueness is still useful in our purpose. Note finally that even such a maximal  $\hat{Q}$  may fail to be equivalent to the reference probability  $\mathbb{P}$ . See [23, Example 2.5] for a counter example. In the sequel, we fix such a maximal dual optimizer, and call  $\hat{P}$  a *least favorable measure*.

The duality (2.10) completely characterizes the *quantitative nature* of the problem (2.6). But our aim in this paper is to discuss the *qualitative nature*, especially the existence of

optimal strategy in  $\Theta_V$ . To do this, assumptions (A1) – (A5) are not enough, and we assume additionally

$$(A6) \quad \sup_{\theta \in \Theta_{bb}} E_P[U(\theta \cdot S_T)] < \infty, \quad \forall P \in \mathcal{P}^e.$$

**Remark 2.3.** Several remarks on assumption (A6) are in order.

1. This assumption is automatically satisfied if  $U(\infty) := \sup_x U(x) < \infty$  as exponential utility, and in this case,  $U(X)^+ \in \bigcap_{P \in \mathcal{P}} L^1(P)$  for any random variable  $X$ . Therefore, the robust utility functional  $X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)]$  is well-defined on  $L^0$  as a  $[-\infty, \infty)$ -valued concave functional.
2. If  $U(\infty) = \infty$ , [2, Th. 1.1 and Remark 1.2] show under (A4) that (A6) is equivalent to:

$$(2.12) \quad \forall P \in \mathcal{P}^e, \exists Q \in \mathcal{M}_V \text{ such that } V(Q|P) < \infty.$$

This is further equivalent to saying that  $v_P(y) < \infty$  for all  $y > 0$  and  $P \in \mathcal{P}^e$ , where  $v_P$  is the dual value function

$$v_P(y) := \inf_{Q \in \mathcal{M}_V} V(yQ|P), \quad y > 0.$$

3. We could state (A6) with the whole  $\mathcal{P}$  rather than  $\mathcal{P}^e$ . But for our purpose, (A6) is enough. Recall that (A5) implies in particular  $\mathcal{P}^e \neq \emptyset$ . If  $\bar{P} \in \mathcal{P}^e$ , we have  $(P + \bar{P})/2 \in \mathcal{P}^e$  for all  $P \in \mathcal{P}$ , and  $\|X\|_{L^1((P+\bar{P})/2)} = (\|X\|_{L^1(P)} + \|X\|_{L^1(\bar{P})})/2 \geq \|X\|_{L^1(P)}/2$ . Hence we have, for instance,

$$(a) \quad \bigcap_{P \in \mathcal{P}} L^1(P) = \bigcap_{P \in \mathcal{P}^e} L^1(P);$$

$$(b) \quad \text{if } (X^n) \text{ is bounded in } L^1(P) \text{ for all } P \in \mathcal{P}^e, \text{ then the same is true for all } P \in \mathcal{P}.$$

In particular, (A6) (hence (2.12)) guarantees even in the case  $U(\infty) = \infty$  that

$$(2.13) \quad X \in \bigcap_{Q \in \mathcal{M}_V} L^1(Q) \Rightarrow U(X)^+ \in \bigcap_{P \in \mathcal{P}} L^1(P).$$

In fact, if  $V(Q|P) < \infty$  and  $X \in L^1(Q)$ , Young's inequality implies  $U(X) \leq V(dQ/dP) + (dQ/dP)X \in L^1(P)$ , and we can take such a  $Q \in \mathcal{M}_V$  by (2.12) for all  $P \in \mathcal{P}^e$ .

**Remark 2.4** (Continuation of Remark 2.1). We give a brief comparison of admissible classes considered in literature. In [17], the class  $\Theta_V(\hat{P})$  is used to discuss the existence of optimal strategy, while [8] considered (implicitly) a slightly smaller class:

$$(2.14) \quad \mathcal{M}_V^0(\hat{Q}, \hat{P}) := \{Q \in \mathcal{M}_{loc} : V(\alpha Q + (1 - \alpha)\hat{Q}|\hat{P}) < \infty, \exists \alpha \in (0, 1)\},$$

$$(2.15) \quad \Theta_V^0(\hat{Q}, \hat{P}) := \left\{ \theta \in L_0(S) : \begin{array}{l} \theta \cdot S \text{ is a } Q\text{-supermartingale,} \\ \forall Q \in \mathcal{M}_V^0(\hat{Q}, \hat{P}) \end{array} \right\}.$$

Note that  $\Theta_V^0(\hat{Q}, \hat{P}) \subset \Theta_V(\hat{P})$  since  $\mathcal{M}_V(\hat{P}) \subset \mathcal{M}_V^0(\hat{Q}, \hat{P})$ , while if we set  $\Theta_m(\hat{Q}) := \{\theta \in L_0(S) : \theta \cdot S \text{ is a } \hat{Q}\text{-martingale}\}$ ,

$$\Theta_V \cap \Theta_m(\hat{Q}) \subset \Theta_V(\hat{P}) \cap \Theta_m(\hat{Q}) = \Theta_V^0(\hat{Q}, \hat{P}) \cap \Theta_m(\hat{Q}).$$

Thus  $\Theta_V(\hat{P})$  and  $\Theta_V^0(\hat{Q}, \hat{P})$  are essentially equivalent for the existence of optimal strategy (see Theorem 3.2). We just emphasize here that our class  $\Theta_V$  depends neither on particular  $P \in \mathcal{P}$  nor  $Q \in \mathcal{M}_V$ , while  $\Theta_V(\hat{P})$  and  $\Theta_V^0(\hat{Q}, \hat{P})$  do.

We conclude this section by recalling a stability property of a set of probability measures, called *m-stability*, which will be used in Theorem 3.2 below.

**Definition 2.5** ([5], Definition 1). A set  $\mathcal{Q}$  of probability measures is said to be *m-stable* (multiplicatively stable) if for any  $Q \in \mathcal{Q}$ ,  $Q' \in \mathcal{Q}^e$  with the density processes  $Z_t = (dQ/d\mathbb{P})|_{\mathcal{F}_t}$  and  $Z'_t = (dQ'/d\mathbb{P})|_{\mathcal{F}_t}$ , as well as any stopping time  $\tau \leq T$ , a new probability  $\tilde{Q}$  defined by  $d\tilde{Q}/d\mathbb{P} := Z_\tau(Z'_\tau/Z'_\tau)$  is an element of  $\mathcal{Q}$ .

This property is equivalent to the *time-consistency* of the corresponding *dynamic coherent monetary utility function*  $\phi_\tau(X) := \text{ess inf}_{Q \in \mathcal{Q}} E_Q[X|\mathcal{F}_\tau]$ : for any  $X, Y \in L^\infty$  and stopping times  $\sigma \leq \tau$ ,  $\phi_\tau(X) \leq \phi_\tau(Y)$  implies  $\phi_\sigma(X) \leq \phi_\sigma(Y)$ . This is further equivalent (under (A3)) to the time-consistency of the dynamic robust utility functional  $\mathcal{U}_\tau(X) := \text{ess inf}_{Q \in \mathcal{Q}} E_Q[U(X)|\mathcal{F}_\tau]$ . See [5, Theorem 12] for details and precise formulation. Note that the set  $\mathcal{M}_{loc}$  of all local martingale measures is m-stable.

### 3. MAIN RESULTS

We first state a result on a “weak solution” to the problem (2.6), which yields a candidate of optimal strategy. Let

$$(3.1) \quad \mathcal{X} := \left\{ X \in L^0 : X \in \bigcap_{Q \in \mathcal{M}_V} L^1(Q), \sup_{Q \in \mathcal{M}_V} E_Q[X] \leq 0 \right\}.$$

Note that  $\theta \cdot S_T \in \mathcal{X}$  if  $\theta \in \Theta_V$ , and  $X \in \mathcal{X}$  implies  $U(x+X)^+ \in \bigcap_{P \in \mathcal{P}} L^1(P)$  for any  $x \in \mathbb{R}$ , by (A6) and Remark 2.3. Thus the robust utility functional  $X \mapsto \inf_{P \in \mathcal{P}} E_P[U(x+X)]$  is well-defined on  $\mathcal{X}$ .

**Theorem 3.1.** Suppose (A1) – (A6), and let  $x \in \mathbb{R}$  and  $(\hat{\lambda}, \hat{Q}, \hat{P})$  be a maximal dual optimizer. Then there exists an  $\hat{X} \in \mathcal{X}$  such that  $U(x+\hat{X}) \in \bigcap_{P \in \mathcal{P}} L^1(P)$  and

$$(3.2) \quad u(x) = \sup_{X \in \mathcal{X}} \inf_{P \in \mathcal{P}} E_P[U(x+X)] = \inf_{P \in \mathcal{P}} E_P[U(x+\hat{X})],$$

where the infimum is attained by  $\hat{P}$ . Moreover, there exists an  $(S, \hat{Q})$ -integrable predictable process  $\hat{\theta}$  with  $\hat{\theta}_0 = 0$  such that  $\hat{\theta} \cdot S$  is a  $\hat{Q}$ -martingale (not only local) and

$$(3.3) \quad x + \hat{X} = -V'(\hat{\lambda} d\hat{Q}/d\hat{P}) = x + \hat{\theta} \cdot S_T, \hat{Q}\text{-a.s.}$$

In particular,  $\hat{X}$  is  $\hat{Q}$ -a.s. unique, and  $\hat{\theta}$  is unique in the sense that  $\hat{\theta} \cdot S$  is unique up to  $\hat{Q}$ -indistinguishability.

The proof is given in Section 4. The first equality in (3.2) states that the robust utility maximization over terminal wealths  $x + \theta \cdot S_T$  is (quantitatively) equivalent to the indirect utility maximization :

$$u_{\mathcal{M}_V}(x) = \sup_{X \in \mathcal{X}} \inf_{P \in \mathcal{P}} E_P[U(x+X)],$$

while the random variable  $\hat{X}$  is the so-called *optimal contingent claim*. Such arguments are quite standard in (non-robust) utility maximization, and also in the robust case, [8, Theorem 3.11] shows a similar result: under (A1) – (A5), the assertions of Theorem 3.1 hold true *except that* the sets  $\mathcal{M}_V$  (in the definition (3.1)) and  $\mathcal{P}$  are replaced by  $\mathcal{M}_V^0(\hat{Q}, \hat{P})$  and  $\mathcal{P}^0(\hat{Q}, \hat{P})$  defined respectively by Remark 2.4 and

$$\mathcal{P}^0(\hat{Q}, \hat{P}) := \{P \in \mathcal{P} : V(\hat{Q}|\alpha P + (1-\alpha)\hat{P}) < \infty, \exists \alpha \in (0, 1)\}.$$

Note that our finite utility assumption (A6) is automatic if  $\mathcal{P}$  is replaced by  $\mathcal{P}^0(\hat{Q}, \hat{P})$ . Also, when  $U(\infty) < \infty$ , the set  $\mathcal{P}^0(\hat{Q}, \hat{P})$  actually coincides with the whole set  $\mathcal{P}$  ([8, Remark 3.10]). However,  $\mathcal{M}_V^0(\hat{Q}, \hat{P})$  still depends on  $(\hat{Q}, \hat{P})$  which is the *solution* to the dual problem, hence not *a priori* available. On the other hand, our formulation is universal, which is a slight, but qualitatively crucial contribution.

Theorem 3.1 suggests that the “strategy”  $\hat{\theta}$  is a candidate of optimal strategy. However, we still have to prove that this strategy is indeed *admissible*.

**Theorem 3.2.** *In addition to (A1) – (A6), we assume that  $\hat{Q} \sim \mathbb{P}$  and  $\mathcal{P}$  is  $m$ -stable. Then  $\hat{\theta}$  is  $(S, \mathbb{P})$ -integrable (hence  $(S, P)$ -integrable for all  $P \in \mathcal{P}$ ), and  $\hat{\theta} \cdot S$  is a supermartingale under all  $Q \in \mathcal{M}_V$ . In particular,  $\hat{\theta}$  belongs to  $\Theta_V$  and is an optimal strategy.*

The proof is given in Section 5. When  $\mathcal{P} = \{\mathbb{P}\}$ , the question of *uniform supermartingale property* of this type goes back to the “six-author paper” [7] which shows that the optimal wealth in *exponential utility* maximization is a *martingale* under all local martingale measures having a finite relative entropy with  $\mathbb{P}$ , under an additional assumption on reverse Hölder inequality which is later removed by [14]. Although this uniform martingale property is no longer true for other utility functions, [21] shows that the optimal wealth is a supermartingale under all  $Q \in \mathcal{M}_V(\mathbb{P})$ , for any utility functions on  $\mathbb{R}$  with reasonable asymptotic elasticity. There are also some extensions to the case where the semimartingale  $S$  is not locally bounded. See e.g. [3] and [4].

In the robust case, the  $Q$ -supermartingale property for all  $Q \in \mathcal{M}_V(\hat{P})$  (hence all  $Q \in \mathcal{M}_V^0(\hat{Q}, \hat{P})$  since  $\hat{\theta} \cdot S$  is a  $\hat{Q}$ -martingale) is shown by [8] (see also [17] for a slight extension). We emphasize that the difference between  $\mathcal{M}_V(\hat{P})$  and  $\mathcal{M}_V$  is essential here. Note that  $\hat{X}$  is also optimal for the utility maximization problem under the fixed measure  $\hat{P}$ , and the same is true for  $(\hat{\lambda}, \hat{Q})$  in the dual side. Thus the result of [21] cited in the previous paragraph is still applicable (under the assumption  $\hat{Q} \sim \mathbb{P}$ ) for  $Q$  with  $V(Q|\hat{P}) < \infty$ , while we have to consider the case where  $V(Q|P) < \infty$  for *some*  $P \in \mathcal{P}$  but possibly  $V(Q|\hat{P}) = \infty$ .

To grasp the situation, we try to describe the heuristics behind the argument in [21] (from our point of view), and our idea of extending it. In what follows in this section, we suppose all the assumptions of Theorem 3.2, especially  $\hat{Q} \sim \mathbb{P}$ .

For a moment, we *suppose* that  $\hat{\theta} \cdot S$  is a  $Q$ -supermartingale for some  $Q \in \mathcal{M}_V$ . Then the  $\hat{Q}$ -martingale property and the representation (3.3) imply: for any stopping time  $\tau \leq T$ ,

$$(3.4) \quad E_{\hat{Q}}[V'(\hat{\lambda} d\hat{Q}/d\hat{P})|\mathcal{F}_{\tau}] \leq E_Q[V'(\hat{\lambda} d\hat{Q}/d\hat{P})|\mathcal{F}_{\tau}], \text{ } Q\text{-a.s.}$$

On the other hand, Ansel-Stricker’s lemma [1] shows that  $\hat{\theta} \cdot S$  is a  $Q$ -supermartingale if and only if there exists a  $Q$ -martingale lower bound, i.e., a  $Q$ -martingale  $M^Q$  such that  $\hat{\theta} \cdot S \geq M^Q$ ,  $Q$ -a.s. In particular, if (3.4) holds true for any stopping time  $\tau \leq T$ , the process defined by  $M_{\tau}^Q = -E_Q[V'(\hat{\lambda} d\hat{Q}/d\hat{P})|\mathcal{F}_{\tau}]$  provides a desired lower bound, hence (3.4) is a necessary and *sufficient* condition for  $\hat{\theta} \cdot S$  to be a  $Q$ -supermartingale.

When  $V(Q|\hat{P}) < \infty$ , the inequality (3.4) is obtained as the *variational inequality* which characterizes  $\hat{Q}$  as a minimizer of the functional  $Q \mapsto V(\hat{\lambda} Q|\hat{P})$  when  $\tau = 0$ , and a “Bellman-type” principle using the  $m$ -stability of the set of local martingale measures shows the case of general  $\tau \leq T$ .

If  $\inf_{P \in \mathcal{P}} V(Q|P) < \infty$  but  $V(Q|\hat{P}) = \infty$ , this argument is no longer applicable at least directly. Mathematically speaking, we lose some important estimates to guarantee the necessary convergences, or more intuitively, any element  $Q$  with  $V(Q|\hat{P}) = \infty$  is in no way optimal at very early stage, and we can not draw further information from the optimality of  $\hat{Q}$  in the minimization of  $Q \mapsto V(\hat{\lambda} Q|\hat{P})$ . However, we have used only a part of information of  $\hat{Q}$  so far, and it is natural to expect that a better information may improve the result. More specifically,



**Step 1** the optimality of  $(\hat{Q}, \hat{P})$  in the minimization of  $(Q, P) \mapsto V(\hat{\lambda}Q|P)$  should yield a variational inequality similar to (3.4) but with an additional term involving  $P$ :

$$“E_{\hat{Q}}[V'(\hat{\lambda}d\hat{Q}/d\hat{P})|\mathcal{F}_\tau] + F_\tau(\hat{P}) \leq E_Q[V'(\hat{\lambda}d\hat{Q}/d\hat{P})|\mathcal{F}_\tau] + F_\tau(P)”.$$

**Step 2** Though we may not take  $P = \hat{P}$  in general, it seems natural to expect that we may take  $P$  “arbitrarily close to  $\hat{P}$ ” keeping  $V(Q|P) < \infty$  with fixed  $Q$ .

**Step 3** If this is the case, we may expect (3.4) by an approximation argument:

$$“F_\tau(P) \rightarrow F_\tau(\hat{P})”.$$

The formal inequality in **Step 1** will be realized as Proposition 5.4 below, where the m-stability of  $\mathcal{P}$  will play an important role. On the other hand, **Steps 2** and **3** will be justified in a certain sense by a simple trick which is a consequence of reasonable asymptotic elasticity (Lemma 5.5).

**Remark 3.3** (What happens when  $\hat{Q} \not\sim \mathbb{P}$ ?). The equivalence  $\hat{Q} \sim \mathbb{P}$  is automatic if all elements of  $\mathcal{P}$  are equivalent to  $\mathbb{P}$ . When the filtration  $\mathbb{F}$  is *continuous* (i.e., every  $(\mathbb{F}, \mathbb{P})$ -martingale is continuous, especially if it is generated by a Brownian motion), the latter condition is already implied by the m-stability of  $\mathcal{P}$  and (A2) (see [5, Theorem 8]), thus it is not a further restriction in that case.

In general, however, the equivalence  $\hat{Q} \sim \mathbb{P}$  may fail (see [23, Example 2.5] for a counter example), thus it is worth asking what happens in that case. When  $U$  is finite only on the positive half-line, the optimal claim  $\hat{X}$  (which does not require the assumption  $\hat{Q} \sim \mathbb{P}$ ) is super-hedged by some  $(S, \mathbb{P})$ -integrable process  $\tilde{\theta}$  with  $\tilde{\theta} \cdot S = \hat{\theta} \cdot S$ ,  $\hat{Q}$ -a.s. By the monotonicity of robust utility functional, we see that  $\tilde{\theta}$  is an optimal strategy without the additional assumption  $\hat{Q} \sim \mathbb{P}$  (see [23] and [22]). However, this argument essentially relies on the fact that  $\hat{X}$  is bounded below by  $-x$  (since  $U(x) = -\infty$  for  $x < 0$ ), and no longer works when the utility function is finite on the entire real line. Thus we can not drop the assumption  $\hat{Q} \sim \mathbb{P}$  (at now).

**Remark 3.4** (Random Endowment). The results of this paper may also be stated with a *random endowment*  $B$  as long as it is an  $\mathcal{F}_T$ -measurable random variable satisfying

$$(3.5) \quad \begin{aligned} &\forall P \in \mathcal{P}, \exists \varepsilon_P > 0 \text{ such that } U(-\varepsilon_P B^+) \in L^1(P), \\ &\exists \varepsilon > 0 \text{ such that } \{U(-(1+\varepsilon)B^-)dP/d\mathbb{P}\}_{P \in \mathcal{P}} \text{ is uniformly integrable.} \end{aligned}$$

Then the robust utility maximization problem (2.6) reads as

$$(3.6) \quad u_B(x) := \sup_{\theta \in \Theta_{bb}} \inf_{P \in \mathcal{P}} E_P[U(x + \theta \cdot S_T + B)],$$

Assumption (3.5) implies that  $B \in \bigcap_{Q \in \mathcal{M}_V} L^1(Q)$ , and guarantees under (A1) – (A5) that a duality corresponding to (2.10) holds true [18, Theorem 2.3]:

$$(3.7) \quad \sup_{\theta \in \Theta_V} \inf_{P \in \mathcal{P}} E_P[U(x + \theta \cdot S_T + B)] = \inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} (V(\lambda Q|\mathcal{P}) + \lambda x + \lambda E_Q[B]),$$

With the same assumptions, the dual problem admits a maximal solution with the unique density in the sense of (2.11). Then Theorems 3.1 and 3.2 remain true with similar proofs, and with obvious modifications, e.g., (3.3) is replaced by  $x + \hat{X} + B = -V'(\hat{\lambda}d\hat{Q}/d\hat{P}) = x + \hat{\theta} \cdot S_T + B$ ,  $\hat{Q}$ -a.s. We omit the details. See [18] for the treatment of random endowment and other implications of (3.5).



#### 4. OPTIMAL CLAIM

We first note that we have only to consider the case  $x = 0$ . Indeed, assumptions (A3) and (A4) on the utility function are invariant under the translation of utility function from  $U$  to  $U_x(\xi) := U(x + \xi)$ , and all the results for  $x \neq 0$  follow from those for  $x = 0$  applied to the new utility function  $U_x$ . Thus we assume  $x = 0$  in what follows.

The next technical lemma is a collection of several arguments in [4].

**Lemma 4.1** ([4]). *Let  $(Q, P)$  be a pair of probabilities with  $V(Q|P) < \infty$ , and  $(k^n)_n$  a sequence of random variables such that  $E_P[U(k^n)]$  is bounded from below and  $E_Q[k^n] \leq 0$  for all  $n$ . Then*

- (a)  $(k^n)_n$  is bounded in  $L^1(Q)$ ;
- (b)  $(U(k^n))_n$  is bounded in  $L^1(P)$ ;
- (c) If in addition  $k^n$  converges a.s. to some  $k \in L^0$ , we have  $k \in L^1(Q)$ ,  $U(k) \in L^1(P)$  and that

$$(4.1) \quad E_Q[k] \leq 0 \text{ and } \limsup_n E_P[U(k^n)] \leq E_P[U(k)].$$

*Proof.* We just fill the gap from [4]. As we are assuming the reasonable asymptotic elasticity (A4), assertions (a) and (b) are contained in Proposition 6.3 of [4]. The assertion (c) also appears (implicitly) in the proof of their Theorem 4.10, which we briefly recall here.

Assume  $k^n \rightarrow k$ ,  $P$ -a.s. Since  $(k^n)$  (resp.  $(U(k^n))$ ) is bounded in  $L^1(Q)$  (resp.  $L^1(P)$ ), Fatou's lemma applied to the sequence  $(|k^n|)_n$  (resp.  $(|U(k^n)|)_n$ ) shows that  $k \in L^1(Q)$  (resp.  $U(k) \in L^1(P)$ ). By Young's inequality, we have  $U(k^n) - \lambda(dQ/dP)k^n \leq V(\lambda dQ/dP) \in L^1(P)$  for all  $n \in \mathbb{N}$  and  $\lambda > 0$ , where the  $P$ -integrability of the right hand side for all  $\lambda$  follows from the reasonable asymptotic elasticity. By this *integrable upper bound* as well as the assumption  $E_Q[k^n] \leq 0$ , (reverse) Fatou's lemma shows that

$$\begin{aligned} \limsup_n E_P[U(k^n)] &\leq \limsup_n E_P[U(k^n) - \lambda(dQ/dP)k^n] \\ &\leq E_P[U(k) - \lambda(dQ/dP)k] = E_P[U(k)] - \lambda E_Q[k], \forall \lambda > 0. \end{aligned}$$

Letting  $\lambda \downarrow 0$ , we have (4.1), while  $E_Q[k] \leq 0$  follows by letting  $\lambda \uparrow \infty$ .  $\square$

*Proof of Theorem 3.1.* We choose a maximizing sequence  $(\theta^n)_n \subset \Theta_{bb}$ , that is

$$(4.2) \quad \inf_{P \in \mathcal{P}} E_P[U(\theta^n \cdot S_T)] \nearrow u(0).$$

This sequence does not have to converge, thus we appeal to a Komlós type argument. Let  $(\bar{Q}, \bar{P}) \in \mathcal{M}_V \times \mathcal{P}$  be such that  $\bar{Q} \sim \bar{P} \sim \mathbb{P}$  and  $V(\bar{Q}|\bar{P}) < \infty$  which exists by (A5). Since  $E_{\bar{P}}[U(\theta^n \cdot S_T)] \geq \inf_{P \in \mathcal{P}} E_P[U(\theta^n \cdot S_T)]$  and  $E_{\bar{Q}}[\theta^n \cdot S_T] \leq 0$  by construction, Lemma 4.1 (a) shows that  $(\theta^n \cdot S_T)_n$  is bounded in  $L^1(\bar{Q})$ . Hence Komlós' theorem (see e.g. [6, Theorem 15.1.3]) yields another sequence  $(\tilde{k}^n)_n$  such that

$$\begin{cases} \tilde{k}^n \in \text{conv}(\theta^n \cdot S_T, \theta^{n+1} \cdot S_T, \dots) \\ \tilde{k}^n \text{ converges } \bar{Q}\text{-a.s. (hence } \mathbb{P}\text{-a.s.) to some } \hat{X} \in L^1(\bar{Q}). \end{cases}$$

By construction, each  $\tilde{k}^n$  is again the terminal value of a stochastic integral  $\tilde{\theta}^n \cdot S_T$  where  $\tilde{\theta}^n$  is the convex combination of  $(\theta^n, \theta^{n+1}, \dots)$  with the same convex weights as  $\tilde{k}^n$ , hence  $\tilde{\theta}^n \in \Theta_{bb}$  and  $E_Q[\tilde{k}^n] \leq 0$  for each  $n$  and  $Q$ , in particular.

Since the robust utility functional  $X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)]$  is concave as a point-wise infimum of concave functionals, we have  $\inf_{P \in \mathcal{P}} E_P[U(\tilde{k}^n)] \geq \inf_{P \in \mathcal{P}} E_P[U(\theta^n \cdot S_T)]$  for each  $n$ . Hence we still have  $\lim_n \inf_{P \in \mathcal{P}} E_P[U(\tilde{k}^n)] = u(0)$ , and the sequence  $(E_P[U(\tilde{k}^n)])_n$  is bounded from below for all  $P \in \mathcal{P}$ .

If  $Q \in \mathcal{M}_V$ , there is a  $P \in \mathcal{P}$  with  $V(Q|P) < \infty$  by the definition of  $\mathcal{M}_V$ , hence another application of Lemma 4.1 to the sequence  $(\tilde{k}^n)$  with the pair  $(Q, P)$  shows that  $\hat{X} \in L^1(Q)$  and  $E_Q[\hat{X}] \leq 0$ . Hence  $\hat{X} \in \mathcal{X}$ .

We next show that  $U(\hat{X}) \in \bigcap_{P \in \mathcal{P}} L^1(P)$  and

$$(4.3) \quad \limsup_n E_P[U(\tilde{k}^n)] \leq E_P[U(\hat{X})], \quad \forall P \in \mathcal{P}.$$

This is immediate from Fatou's lemma if  $U$  is bounded from above. When  $U(\infty) = \infty$  and  $P \in \mathcal{P}^e$ , we can take a  $Q \in \mathcal{M}_V$  with  $V(Q|P) < \infty$  by (2.12), hence Lemma 4.1 shows (4.3) and that  $(U(\tilde{k}^n))_n$  is bounded in  $L^1(P)$ . Then Remark 2.3 shows that  $U(\hat{X}) \in L^1(P)$  and  $(U(\tilde{k}^n))_n$  is still bounded in  $L^1(P)$  for arbitrary  $P \in \mathcal{P}$  which need not be equivalent to  $\mathbb{P}$ . To prove (4.3) in the case  $P \not\sim \mathbb{P}$ , we take  $(\bar{Q}, \bar{P})$  as above, and set  $P_\alpha := \alpha P + (1 - \alpha)\bar{P}$  for  $\alpha \in (0, 1)$ . Since  $P_\alpha \sim \mathbb{P}$ , the claim is true for  $P_\alpha$  for all  $\alpha \in (0, 1)$ , while we see that  $\sup_n |E_{P_\alpha}[U(\tilde{k}^n)] - E_P[U(\tilde{k}^n)]| \leq 2(1 - \alpha) \sup_n (\|U(\tilde{k}^n)\|_{L^1(P)} \vee \|U(\tilde{k}^n)\|_{L^1(\bar{P})}) \rightarrow 0$ , as  $\alpha \uparrow 1$ . Thus we deduce

$$\begin{aligned} \limsup_n E_P[U(\tilde{k}^n)] &= \lim_{\alpha \uparrow 1} \limsup_n E_{\alpha P + (1 - \alpha)\bar{P}}[U(\tilde{k}^n)] \\ &\leq \lim_{\alpha \uparrow 1} E_{\alpha P + (1 - \alpha)\bar{P}}[U(\hat{X})] = E_P[U(\hat{X})]. \end{aligned}$$

Hence (4.3) holds for all  $P \in \mathcal{P}$ .

We now prove (3.2). Note first that for all  $\lambda > 0$ ,  $X \in \mathcal{X}$ ,  $Q \in \mathcal{M}_V$  and  $P \in \mathcal{P}$ ,

$$E_P[U(X)] \leq V(\lambda Q|P) + \lambda E_Q[X] \leq V(\lambda Q|P).$$

In particular,

$$\inf_{P \in \mathcal{P}} E_P[U(X)] \leq \inf_{\lambda > 0} \inf_{(Q, P) \in \mathcal{M}_V} V(\lambda Q|P) \stackrel{(2.10)}{=} u(0), \quad \forall X \in \mathcal{X},$$

On the other hand, (4.3) shows

$$u(0) = \lim_n \inf_{P \in \mathcal{P}} E_P[U(\tilde{k}^n)] \leq \inf_{P \in \mathcal{P}} \limsup_n E_P[U(\tilde{k}^n)] \leq \inf_{P \in \mathcal{P}} E_P[U(\hat{X})].$$

This concludes the proof of (3.2).

We proceed to (3.3). Notice that

$$(4.4) \quad U(\hat{X}) = V(\hat{\lambda} d\hat{Q}/d\hat{P}) + \hat{\lambda} (d\hat{Q}/d\hat{P})\hat{X}, \quad \hat{P}\text{-a.s.}$$

Indeed, “ $\leq$ ” is just a Young's inequality, while “ $\geq$ ” follows from

$$\begin{aligned} u(0) &= \inf_{P \in \mathcal{P}} E_P[U(\hat{X})] \leq E_{\hat{P}}[U(\hat{X})] \stackrel{(i)}{\leq} E_{\hat{P}} \left[ V \left( \hat{\lambda} \frac{d\hat{Q}}{d\hat{P}} \right) + \hat{\lambda} \frac{d\hat{Q}}{d\hat{P}} \hat{X} \right] \\ &\stackrel{(ii)}{\leq} V(\hat{\lambda} \hat{Q}|\hat{P}) \stackrel{(2.10)}{=} u(0). \end{aligned}$$

Here (i) follows from the “ $\leq$ ” part, and (ii) from  $\hat{X} \in \mathcal{X}$ . In particular,  $\hat{P}$  attains the infimum in (3.2) and we obtain (4.4). But an elementary knowledge from convex analysis shows that this is possible only if

$$\hat{X} = -V'(\hat{\lambda} d\hat{Q}/d\hat{P}), \quad \hat{P}\text{-a.s.}$$

This is the first equality in (3.3), and the  $\hat{Q}$ -a.s. uniqueness of  $\hat{X}$  follows from that of  $\hat{\lambda} d\hat{Q}/d\hat{P}$  (see (2.11)). On the other hand, the existence of  $\hat{\theta} \in L(S, \hat{Q})$  with  $\theta_0 = 0$  and  $\hat{\theta} \cdot S$  being a  $\hat{Q}$ -martingale, which represents  $-V'(\hat{\lambda} d\hat{Q}/d\hat{P})$  as (3.3), follows from Theorem 3.2 of [11] (see also [20, Theorem 2.2 (iv)]). Finally,  $\hat{Q}$ -a.s. uniqueness of the process  $\hat{\theta} \cdot S$  follows from the  $\hat{Q}$ -a.s. uniqueness of the terminal value  $\hat{\theta} \cdot S_T$  and the fact that  $\hat{\theta} \cdot S$  is a  $\hat{Q}$ -martingale.  $\square$

### 5. UNIFORM SUPERMARTINGALE PROPERTY OF OPTIMAL WEALTH

We now proceed to the uniform supermartingale property of the optimal wealth, that is, we shall show that  $\hat{\theta} \cdot S$  is a supermartingale under all local martingale measures  $Q$  with finite entropy w.r.t. some  $P \in \mathcal{P}$ . As outlined in Section 3, this will follow if we can prove the dynamic variational inequality (3.4) for every  $Q \in \mathcal{M}_V$ . Therefore, the key of this section is the next proposition which should be compared with [8, Lemma 3.12]. Recall that we have only to consider the case  $x = 0$ . *In what follows, all the assumptions of Theorem 3.2 are in force, and we do not cite them in each statement.*

**Proposition 5.1.** *We have*

1. *for all  $Q \in \mathcal{M}_V$ , and for all stopping time  $\tau \leq T$ ,*

$$(5.1) \quad E_Q \left[ V' \left( \hat{\lambda} d\hat{Q}/d\hat{P} \right) \middle| \mathcal{F}_\tau \right] \geq E_{\hat{Q}} \left[ V' \left( \hat{\lambda} d\hat{Q}/d\hat{P} \right) \middle| \mathcal{F}_\tau \right], \text{ } Q\text{-a.s.}$$

2. *for all  $P \in \mathcal{P}$ , and for all stopping time  $\tau \leq T$ ,*

$$(5.2) \quad E_P[U(\hat{\theta} \cdot S_T) | \mathcal{F}_\tau] \geq E_{\hat{P}}[U(\hat{\theta} \cdot S_T) | \mathcal{F}_\tau], \text{ } P\text{-a.s.}$$

We introduce some notations. If  $L$  is a strictly positive martingale, we denote  $L_{\tau,T} := L_T/L_\tau$ , for any stopping time  $\tau \leq T$ . Recall that any probability  $Q \ll \mathbb{P}$  is identified with a (uniformly integrable) martingale, namely the *density process*  $Z^Q = (dQ/d\mathbb{P})|_{\mathcal{F}_\cdot}$ . In what follows, we denote by  $\hat{Z}$  (resp.  $\hat{D}$ ) the density process of  $\hat{Q}$  (resp.  $\hat{P}$ ). Also, when a pair  $(Q, P) \in \mathcal{M}_{loc} \times \mathcal{P}$  is fixed, the density process of  $Q$  (resp.  $P$ ) is denoted by  $Z$  (resp.  $D$ ), and set:

$$(5.3) \quad Z_{\tau,T}^\alpha := \alpha Z_{\tau,T} + (1-\alpha)\hat{Z}_{\tau,T}, \quad D_{\tau,T}^\alpha := \alpha D_{\tau,T} + (1-\alpha)\hat{D}_{\tau,T}, \quad \alpha \in [0, 1].$$

We make a couple of simple reductions. The first one is just a notational reduction. In our purpose, we can assume without loss of generality that  $\hat{\lambda} = 1$  *since we already know*  $\hat{\lambda}$ . Indeed,  $(\hat{\lambda}\hat{Q}, \hat{P})$  minimizes  $(v, P) \mapsto V(v|P)$  if and only if  $(\hat{Q}, \hat{P})$  minimizes  $(v, P) \mapsto V_{\hat{\lambda}}(v|P) := \frac{1}{\hat{\lambda}} V(\hat{\lambda}v|P)$ . Next, we have only to prove (5.1) and (5.2) for all  $Q \in \mathcal{M}_V^e$  and  $P \in \mathcal{P}^e$ , respectively. Indeed, if we could show (5.1) for all  $Q' \in \mathcal{M}_V^e$  for instance, we have  $\bar{Q} := (Q + \hat{Q})/2 \in \mathcal{M}_V^e$  for any  $Q \in \mathcal{M}_V$  on the one hand, and on the other hand, Bayes' formula implies

$$\begin{aligned} E_{\hat{Q}}[\Phi | \mathcal{F}_\tau] &\leq E_{\bar{Q}}[\Phi | \mathcal{F}_\tau] \\ &= \frac{Z_\tau}{Z_\tau + \hat{Z}_\tau} E_Q[\Phi | \mathcal{F}_\tau] + \frac{\hat{Z}_\tau}{Z_\tau + \hat{Z}_\tau} E_{\hat{Q}}[\Phi | \mathcal{F}_\tau] \text{ a.s. on } \{Z_\tau > 0\} \end{aligned}$$

where  $\Phi = V'(d\hat{Q}/d\hat{P})$ , hence (5.1). A similar argument applies also to (5.2).

The first step is to show a ‘‘Bellman-type’’ principle for a *time-consistent* optimization. Note that the set  $\mathcal{M}_{loc}$  of all local martingale measures is m-stable, while  $\mathcal{M}_V$  is not. The next simple lemma allows us to avoid this difficulty.

**Lemma 5.2.** *Let  $(Q, P) \in \mathcal{M}_V^e \times \mathcal{P}^e$  with  $V(Q|P) < \infty$ , and  $(Z, D)$  the corresponding density processes as well as  $\alpha \in [0, 1]$ . Then for any stopping time  $\tau \leq T$ , the random variable  $\hat{D}_\tau D_{\tau,T}^\alpha V \left( \frac{\hat{Z}_\tau Z_{\tau,T}^\alpha}{\hat{D}_\tau D_{\tau,T}^\alpha} \right)$  is  $\mathcal{F}_\tau$ -locally integrable i.e., there exists an increasing sequence  $A_n \in \mathcal{F}_\tau$  such that*

$$(5.4) \quad \mathbb{P}(A_n) \nearrow 1 \quad \text{and} \quad 1_{A_n} \hat{D}_\tau D_{\tau,T}^\alpha V \left( \frac{\hat{Z}_\tau Z_{\tau,T}^\alpha}{\hat{D}_\tau D_{\tau,T}^\alpha} \right) \in L^1, \forall n.$$

*Proof.* Since  $\hat{D}_\tau D_{\tau,T}^\alpha V \left( \frac{\hat{Z}_\tau Z_{\tau,T}^\alpha}{\hat{D}_\tau D_{\tau,T}^\alpha} \right) \leq \alpha \hat{D}_\tau D_{\tau,T} V \left( \frac{\hat{Z}_\tau Z_{\tau,T}}{\hat{D}_\tau D_{\tau,T}} \right) + (1-\alpha) \hat{D}_T V \left( \frac{\hat{Z}_T}{\hat{D}_T} \right)$  (see the proof of Lemma 5.5 below), and the second term is integrable, it suffices to prove the case  $\alpha = 1$ .

Recall from [10] that the condition (A4) of reasonable asymptotic elasticity is equivalent to: for any  $a \geq 1$ , there exists  $C_a, C'_a > 0$  such that

$$(5.5) \quad V(\lambda y) \leq C_a V(y) + C'_a(y+1), \quad \forall \lambda \in [a^{-1}, a], \forall y > 0.$$

Since  $V$  is bounded from below by  $U(0)$ , we can choose the constant  $C'_a$  so that the right hand side is always positive. For the sequence  $A_n$ , we take

$$A_n := \{\hat{Z}_\tau, Z_\tau, \hat{D}_\tau, D_\tau \in (n^{-1}, n)\} \in \mathcal{F}_\tau, \quad \forall n.$$

Noting that  $\varphi := \hat{D}_\tau D_{\tau,T} V \left( \frac{\hat{Z}_\tau Z_{\tau,T}}{\hat{D}_\tau D_{\tau,T}} \right) = \frac{\hat{D}_\tau}{D_\tau} D_T V \left( \frac{\hat{Z}_\tau Z_{\tau,T}}{\hat{D}_\tau Z_\tau} \frac{Z_T}{D_T} \right)$ , (5.5) implies that

$$\varphi \leq n^2 C_{n^4} D_T V(Z_T/D_T) + n^2 C'_{n^4} (Z_T + D_T) \text{ a.s. on } A_n.$$

Thus  $1_{A_n} \varphi \in L^1$  for each  $n$ . Finally,  $\mathbb{P}(A_n) \nearrow 1$  since  $\hat{Q} \sim \hat{P} \sim Q \sim P \sim \mathbb{P}$  by assumption.  $\square$

**Lemma 5.3.** *For any  $(Q, P) \simeq (Z, D) \in \mathcal{M}_V^e \times \mathcal{P}^e$  with  $V(Q|P) < \infty$ ,  $\alpha \in [0, 1]$ ,*

$$(5.6) \quad E \left[ \hat{Z}_T V \left( \frac{\hat{Z}_T}{\hat{D}_T} \right) \mid \mathcal{F}_\tau \right] \leq E \left[ \hat{D}_\tau D_{\tau,T}^\alpha V \left( \frac{\hat{Z}_\tau Z_{\tau,T}^\alpha}{\hat{D}_\tau D_{\tau,T}^\alpha} \right) \mid \mathcal{F}_\tau \right] \text{ a.s.}$$

*Proof.* Note first that the conditional expectation of the right hand side is well-defined and a.s. finite by Lemma 5.2. Let  $C'$  be the set on which the inequality (5.6) fails, which is  $\mathcal{F}_\tau$ -measurable. Then we suppose by way of contradiction that  $\mathbb{P}(C') > 0$ .

Take a sequence  $(A_n) \subset \mathcal{F}_\tau$  as in Lemma 5.2 and a large  $n$  so that  $\mathbb{P}(C' \cap A_n) > 0$ . Setting  $C := C' \cap A_n$ , we define a new pair  $(\bar{Q}, \bar{P}) \simeq (\bar{Z}, \bar{D})$  by

$$\bar{Z}_T = 1_{C^c} \hat{Z}_T + 1_C \hat{Z}_\tau Z_{\tau,T}^\alpha \text{ and } \bar{D}_T = 1_{C^c} \hat{D}_T + 1_C \hat{D}_\tau D_{\tau,T}^\alpha.$$

First,  $(\bar{Q}, \bar{P}) \in \mathcal{M}_{loc} \times \mathcal{P}$  by the m-stability of  $\mathcal{M}_{loc}$  and  $\mathcal{P}$ . Also, since

$$\bar{D}_T V \left( \frac{\bar{Z}_T}{\bar{D}_T} \right) = 1_{C^c} \hat{D}_T V \left( \frac{\hat{Z}_T}{\hat{D}_T} \right) + 1_C \hat{D}_\tau D_{\tau,T}^\alpha V \left( \frac{\hat{Z}_\tau Z_{\tau,T}^\alpha}{\hat{D}_\tau D_{\tau,T}^\alpha} \right),$$

we have  $V(\bar{Q}|\bar{P}) < \infty$  by the construction of  $C$  and Lemma 5.2, hence  $\bar{Q} \in \mathcal{M}_V$ . Finally,

$$\begin{aligned} V(\bar{Q}|\bar{P}) &= E \left[ \bar{D}_T V \left( \frac{\bar{Z}_T}{\bar{D}_T} \right) \right] \\ &= E \left[ 1_{C^c} E \left[ \hat{D}_T V \left( \frac{\hat{Z}_T}{\hat{D}_T} \right) \mid \mathcal{F}_\tau \right] + 1_C E \left[ \hat{D}_\tau D_{\tau,T}^\alpha V \left( \frac{\hat{Z}_\tau Z_{\tau,T}^\alpha}{\hat{D}_\tau D_{\tau,T}^\alpha} \right) \mid \mathcal{F}_\tau \right] \right] \\ &< V(\hat{Q}|\hat{P}). \end{aligned}$$

This contradict to the optimality of  $(\hat{Q}, \hat{P})$ .  $\square$

Now the formal inequality in **Step 1** at the end of Section 3 is realized as follows.

**Proposition 5.4.** *For any  $(Q, P) \simeq (Z, D) \in \mathcal{M}_V^e \times \mathcal{P}^e$  with  $V(Q|P) < \infty$ ,*

$$(5.7) \quad \begin{aligned} &\hat{Z}_\tau \left\{ E_Q \left[ V'(d\hat{Q}/d\hat{P}) \mid \mathcal{F}_\tau \right] - E_{\hat{Q}} \left[ V'(d\hat{Q}/d\hat{P}) \mid \mathcal{F}_\tau \right] \right\} \\ &\quad + \hat{D}_\tau \left\{ E_P[U(\hat{X})|\mathcal{F}_\tau] - E_{\hat{P}}[U(\hat{X})|\mathcal{F}_\tau] \right\} \geq 0, \text{ a.s.} \end{aligned}$$

*Proof.* Let  $(Z, D)$ ,  $\tau$ ,  $\alpha$  be as above, and set

$$G_\tau(\alpha) := \hat{D}_\tau D_{\tau,T}^\alpha V(\hat{Z}_\tau Z_{\tau,T}^\alpha / \hat{D}_\tau D_{\tau,T}^\alpha).$$

Then  $\alpha \mapsto G_\tau(\alpha)$  is convex (a.s.) by (the proof of) Lemma 5.5 below, hence  $(G_\tau(\alpha) - G(0))/\alpha$  decreases a.s. to the limit  $\Xi_\tau(Q, P)$  as  $\alpha \searrow 0$ . Here  $\Xi_\tau(Q, P)$  is explicitly computed as:

$$\Xi_\tau(Q, P) = \hat{Z}_\tau V' \left( \frac{d\hat{Q}}{d\hat{P}} \right) (Z_{\tau,T} - \hat{Z}_{\tau,T}) + \hat{D}_\tau U(\hat{X})(D_{\tau,T} - \hat{D}_{\tau,T}),$$

using  $\hat{Z}_T/\hat{D}_T = d\hat{Q}/d\hat{P}$  and  $U(\hat{X}) = V(d\hat{Q}/d\hat{P}) - (d\hat{Q}/d\hat{P})V'(d\hat{Q}/d\hat{P})$ . Since  $G_\tau(1)$  is  $\mathcal{F}_\tau$ -locally integrable and  $E[(G_\tau(\alpha) - G_\tau(0))/\alpha | \mathcal{F}_\tau] \geq 0$  a.s. by Lemma 5.3, the (generalized) conditional monotone convergence theorem shows that  $E[\Xi(Q, P) | \mathcal{F}_\tau] \geq 0$ . Noting that  $V'(d\hat{Q}/d\hat{P}) = -\hat{X} \in L^1(Q)$  and  $U(\hat{X}) \in L^1(P)$  by Theorem 3.1, we deduce (5.7) from Bayes' formula.  $\square$

We proceed to **Step 2**. Fixing  $Q \in \mathcal{M}_V$ , we want to take  $P$  “arbitrarily close” to  $\hat{P}$ . The next simple lemma gives a precise form of this argument.

**Lemma 5.5.** *Let  $(Q, P)$  and  $(Q', P')$  be any two pairs of probability measures absolutely continuous w.r.t.  $\mathbb{P}$ . Then for any  $\alpha, \gamma \in (0, 1)$ , we have*

$$(5.8) \quad \begin{aligned} & V(\alpha Q + (1 - \alpha)Q' | \gamma P + (1 - \gamma)P') \\ & \leq \gamma V \left( \frac{\alpha}{\gamma} Q \mid P \right) + (1 - \gamma) V \left( \frac{1 - \alpha}{1 - \gamma} Q' \mid P' \right). \end{aligned}$$

In particular,  $V(Q|P) < \infty$  and  $V(Q'|P') < \infty$  imply  $V(\alpha Q + (1 - \alpha)Q' | \gamma P + (1 - \gamma)P') < \infty$  for any  $\alpha, \gamma \in (0, 1)$ .

*Proof.* Note that for any positive numbers  $x, x', y, y'$ ,

$$\frac{\alpha x + (1 - \alpha)x'}{\gamma y + (1 - \gamma)y'} = \frac{\gamma y}{\gamma y + (1 - \gamma)y'} \frac{\alpha x}{\gamma y} + \frac{(1 - \gamma)y'}{\gamma y + (1 - \gamma)y'} \frac{1 - \alpha}{1 - \gamma} \frac{x'}{y'}.$$

Thus the convexity of  $V$  shows that

$$\begin{aligned} & (\gamma y + (1 - \gamma)y') V \left( \frac{\alpha x + (1 - \alpha)x'}{\gamma y + (1 - \gamma)y'} \right) \\ & \leq \gamma V \left( \frac{\alpha x}{\gamma y} \right) + (1 - \gamma) V \left( \frac{1 - \alpha}{1 - \gamma} \frac{x'}{y'} \right). \end{aligned}$$

Putting  $dQ/d\mathbb{P}$  (resp.  $dQ'/d\mathbb{P}$ ,  $dP/d\mathbb{P}$ ,  $dP'/d\mathbb{P}$ ) into  $x$  (resp.  $x'$ ,  $y$ ,  $y'$ ), and taking the  $\mathbb{P}$ -expectation, this implies (5.8). The second claim follows from the fact that  $V(Q|P) < \infty \Rightarrow V(\lambda Q|P) < \infty$  for any  $\lambda > 0$ , as a consequence of reasonable asymptotic elasticity.  $\square$

*Proof of Proposition 5.1.* As noted after the statement of Proposition 5.1, we have only to consider the case  $(Q, P) \in \mathcal{M}_V^e \times \mathcal{P}^e$  with  $V(Q|P) < \infty$ . Fixing such a pair  $(Q, P)$ , we put  $Q_\alpha := \alpha Q + (1 - \alpha)\hat{Q}$  and  $P_\gamma := \gamma P + (1 - \gamma)\hat{P}$  for any  $\alpha, \gamma \in (0, 1)$ . By Lemma 5.5, the auxiliary variational inequality (5.7) is valid for any  $(Q_\alpha, P_\gamma)$  with arbitrary  $\alpha, \gamma \in (0, 1)$ . Noting that  $E_{Q_\alpha}[\Phi | \mathcal{F}_\tau] - E_{\hat{Q}}[\Phi | \mathcal{F}_\tau] = \frac{\alpha Z_\tau}{\alpha Z_\tau + (1 - \alpha)\hat{Z}_\tau} \{E_Q[\Phi | \mathcal{F}_\tau] - E_{\hat{Q}}[\Phi | \mathcal{F}_\tau]\}$  etc, we have

$$\begin{aligned} & \hat{Z}_\tau \frac{\alpha Z_\tau}{\alpha Z_\tau + (1 - \alpha)\hat{Z}_\tau} \{E_Q[V'(d\hat{Q}/d\hat{P}) | \mathcal{F}_\tau] - E_{\hat{Q}}[V'(d\hat{Q}/d\hat{P}) | \mathcal{F}_\tau]\} \\ & + \hat{D}_\tau \frac{\gamma D_\tau}{\gamma D_\tau + (1 - \gamma)\hat{D}_\tau} \{E_P[U(\hat{X}) | \mathcal{F}_\tau] - E_{\hat{P}}[U(\hat{X}) | \mathcal{F}_\tau]\} \\ & \geq 0, \text{ a.s. } \forall \alpha, \gamma \in (0, 1). \end{aligned}$$

Since  $\gamma D_\tau / (\gamma D_\tau + (1 - \gamma) \hat{D}_\tau) \xrightarrow{\gamma \downarrow 0} 0$  and  $\alpha Z_\tau / (\alpha Z_\tau + (1 - \alpha) \hat{Z}_\tau) \xrightarrow{\alpha \downarrow 0} 0$ , we deduce (5.1) and (5.2) by letting  $\gamma \downarrow 0$  (resp.  $\alpha \downarrow 0$ ) with  $\alpha$  (resp.  $\gamma$ ) being fixed, whenever  $V(Q|P) < \infty$ . Finally, any  $Q \in \mathcal{M}_V^e$  (resp.  $P \in \mathcal{P}^e$ ) admits a  $P \in \mathcal{P}$  (resp.  $Q \in \mathcal{M}_V$ ) with  $V(Q|P) < \infty$  by definition (resp. by Remark 2.3).  $\square$

*Proof of Theorem 3.2.* Under the assumption  $\hat{Q} \sim \mathbb{P}$ , the  $(S, \mathbb{P})$ -integrability of  $\hat{\theta}$  is clear. We verify that  $\hat{\theta} \cdot S$  is a supermartingale under each  $Q \in \mathcal{M}_V$ . Since  $V'(d\hat{Q}/d\hat{P}) \in L^1(Q)$ , the process defined by  $M_\tau^Q = -E_Q[V'(d\hat{Q}/d\hat{P})|\mathcal{F}_\tau]$  is a  $Q$ -martingale. Then (3.3), (5.1) as well as the fact that  $\hat{\theta} \cdot S$  is a  $\hat{Q}$ -martingale show that

$$\hat{\theta} \cdot S_\tau = -E_{\hat{Q}}[V'(d\hat{Q}/d\hat{P})|\mathcal{F}_\tau] \geq M_\tau^Q, \quad Q\text{-a.s.}$$

for any stopping time  $\tau \leq T$ . A stochastic integral w.r.t. a  $Q$ -local martingale dominated below by a  $Q$ -(uniformly integrable) martingale is a  $Q$ -supermartingale by [24, Theorem 1], which is a variant of Ansel-Stricker's lemma [1, Proposition 3.3].  $\square$

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